$$\int \frac{dx}{\sqrt{e^{2x}-1}}$$

$u = x^5+1$ $du = 5x^4 dx$

constant

$\ln|x^5+1| \cdot (5x^4)$

e) $\frac{d}{dy}\left[\int_{y^2}^{2y} e^{3x}\,dx\right]$

$u = 2y$ $du = 2dy$

$u = y^2$ $du = 2y$

$e^{3(2y)} \cdot 2 - e^{3(y^2)}(2y)$

$2e^{6y} - 2ye^{3y^2}$

c) $\frac{d}{dw}\left[\int_w^2 \frac{3}{x}\,dx\right] = 0 - \frac{3}{w}\cdot 1$

constant

$x = w$

$dx = dw = 1$

c) If $f(x) = \int_0^{2x^2} \sqrt{t^2+1}\;dt$, then $f'(-1)$ is __________.

$F'(x)$ $\frac{d}{dx}\left[\int_0^{2x^2} \sqrt{T^2+1}\,dT\right] = F'(x)$

$T = 2x^2$

$dT = 4x$

$F'(x) = \sqrt{(2x^2)^2+1} \cdot 4x = 4x\sqrt{4x^4+1}$

$F'(-1) = 4(-1)\sqrt{4(-1)^4+1} = -4\sqrt{4+1} = -4\sqrt{5}$

3. An object in rectilinear motion is moving along a horizontal line with velocity $v(t) = 3t^2 - 6t$ (in meters per second). If at time $t = 1$, the object is 2 m from the origin, what is its position at $t = 4$.

$V(T) = 3T^2 - 6T = 3T(T-2)$

Stopped $T=0$ and $T=2$

$S(T) = \int V(T)\,dT = \int (3T^2 - 6T)\,dT = T^3 - 3T^2 + C$

$S(T) = T^3 - 3T^2 + C$

$S(T) = 2 = 1^3 - 3(1)^2 + C = 1 - 3 + C = 2$ $C = 4$

$S(T) = T^3 - 3T^2 + 4 \Rightarrow S(4) = 4^3 - 3(4)^2 + 4 = 64 - 48 + 4 = 20$

$S(4) = 2 + \int_1^4 (3T^2 - 6T)\,dT = 2 + T^3 - 3T^2 \Big|_1^4 = 2 + \left[4^3 - 3(4)^2 - (1^3 - 3(1)^2)\right]$

2+64−48−1+3=20

4. An investment in a hedge fund is growing at a continuous rate of $H'(t) = 1105.17(1.105)^t$ dollars per year. Initially \$1000 is invested.
(Calculator may be used)

a) Interpret $\int_0^{10} H'(t)\ dt$ in the context of the problem.

The amount after 10 years
Earned

b) Write an expression (with integrals) that models the amount of money in the hedge fund after t years.

$$\int_0^T 1105.17(1.105)^T dT$$

c) Find the amount of dollars after 10 years.

$$\int_0^{10} 1105.17(1.105)^t dt$$

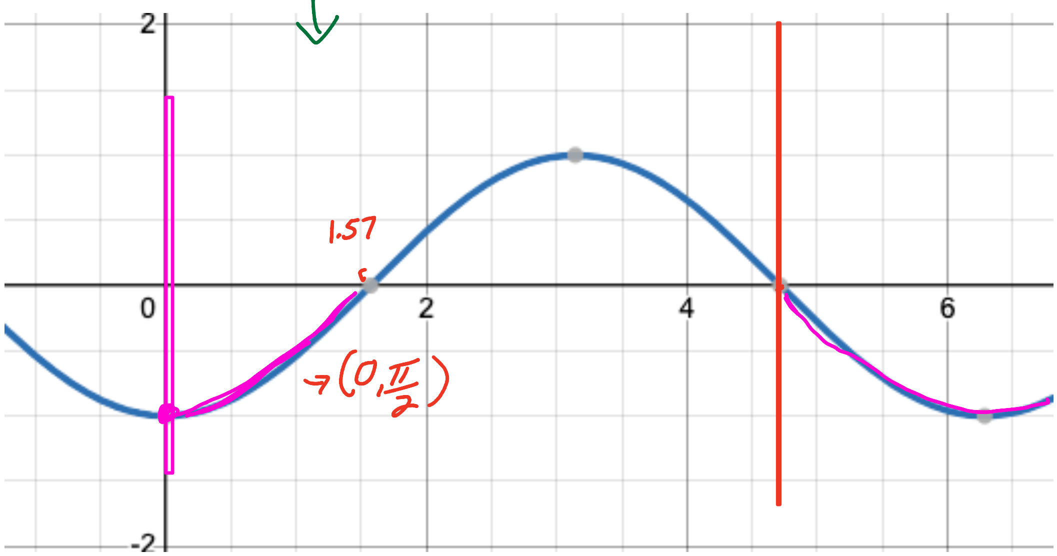
$= 18972.8516592$

5. Suppose $g(x) = \int_0^x \sin\left(t - \frac{\pi}{2}\right)\, dt$ for $0 \le t \le \frac{3\pi}{2}$. On which interval is g is decreasing.

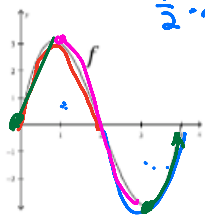
$g'(x) = -$ when g is decreasing

$$g'(x) = \frac{d}{dx}\int_0^x \sin\left(T - \frac{\pi}{2}\right) dT$$

$$g'(x) = \sin\left(x - \frac{\pi}{2}\right)$$



6. The graph of f is given below.



$\frac{1}{2} \cdot 2 \cdot 3 = 3$

$g(x) = \int_0^x F(T)\,dT$

$\frac{d}{dx}[g(x)] = \frac{d}{dx}\int_0^x F(T)\,dT$

$g'(x) = F(x)$

$g'(x) = F(x)$

Let $g(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 4$.

a) When is g increasing? Decreasing? Justify your answer. Find the x-value of the relative extremum of g.

$g'(x) = + = F(x)$ $\quad$ $g'(x) = - = F(x)$

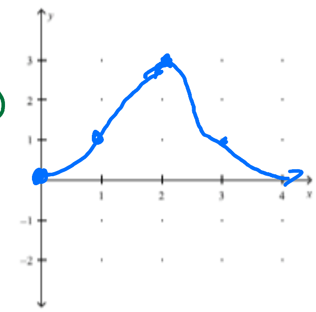
$(0,2)$ $\quad$ $(2,4)$

b) When is g concave up? Concave down? Justify your answer.

$g''(x) = + = F'(x) = \text{Slope of } F(x)$ $\quad (0,1) \cup (3,4)$

$g''(x) = - = F'(x) = \text{Slope of } F(x) = (1,3)$

c) Use the information above to sketch g on the axis provided (for $0 \le x \le 4$)



$$\int (x+2)^2\,dx =$$

$\int (x^2 + 4x + 4)\,dx$

$\frac{1}{3}x^3 + 2x^2 + 4x + C$

$U = x+2$

$dU = dx$

$$\int (x+2)^{100}\,dx =$$

$\int U^{100} \cdot dU = 1 \cdot \frac{1}{101} U^{101} + C$ $\quad 100+1=101$

$\frac{1}{101}(x+2)^{101} + C$

Outside function

Composition of Functions

$$\int f'(\underbrace{g(x)}_{u})\underbrace{g'(x)dx}_{du} = f(\underbrace{g(x)}_{u})+C$$

inside function

Derivative of inside function

$$u = g(x)$$
$$du = g'(x)dx$$

$$\int f'(u)du = f(u)+C$$

Consider this "Anti-Chain Rule"

When deciding what to use for *u*, choose functions in this order: **LIPET**
Logs,
Inverse Trig,
Polynomials,
Exponential,
Trig

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1) Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2) Compute $du = g'(x)dx$.
3) Rewrite the integral in terms of the variable u.
4) Find the resulting integral in terms of u.
5) Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6) Check your answer by differentiating.

$$\int (\underline{x+2})^5\,dx = \frac{1}{6}(x+2)^6 + C$$

$u = x+2$
$du = dx$

$$\int u^5\,du$$

$$\frac{1}{6}u^{5+1=6} + C$$

$$\frac{u^6}{6} + C = \frac{(x+2)^6}{6} + C$$

$$\int \sqrt{1+x^2}\cdot 2x\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$\boxed{\frac{du}{2x} = dx}$$

$$\int \sqrt{u}\cdot \cancel{2x}\cdot \frac{du}{\cancel{2x}} = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}+1=\frac{3}{2}} + C = \frac{2(u^{3/2})}{3} + C$$

$$\frac{2}{3}(1+x^2)^{3/2} + C$$

$$\frac{2(1+x^2)(\sqrt{1+x^2})}{3} + C$$

$$\boxed{\int 3x\,dx = 3\int x\,dx}$$

$$\int \sqrt{4x-1}\,dx$$

$u = 4x-1$

$du = 4\,dx$

$\frac{du}{4} = dx$

$$\int u^{\frac{1}{2}}\frac{du}{4} = \frac{1}{4}\int u^{\frac{1}{2}}\,du$$

$$\frac{1}{4}\cdot\frac{2}{3}\cdot u^{\frac{1}{2}+1=\frac{3}{2}} + C = \frac{2}{12}u^{3/2} + C = \frac{1}{6}(4x-1)^{3/2} + C$$

$$\frac{1}{6}(4x-1)\sqrt{4x-1} + C$$

$$\int \cos(7x+5)\,dx$$

$u = 7x+5$

$du = 7\,dx$

$\frac{du}{7} = dx$

$$\int \cos u \cdot \frac{du}{7}$$

$$\int \frac{1}{7}\cos u\,du = \frac{1}{7}\sin u + C$$

$$\frac{1}{7}\sin(7x+5) + C$$

$$\int \frac{1}{7}\cos u\,du$$

$$\frac{1}{7}\int \cos u\,du$$

$$\frac{1}{7}\sin u + C = \frac{1}{7}\sin(7x+5) + C$$

$$\int x^2 \sin\left(x^3\right)\, dx$$

$u = x^3$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

$\int \cancel{x^2} \sin u \cdot \frac{du}{3\cancel{x^2}} = \int \frac{1}{3} \sin u\, du$

$\frac{1}{3} \int \sin u\, du$

$\frac{1}{3}(-\cos u) + C$

$-\frac{1}{3} \cos x^3 + C$

$$\int \sin^4 x \cdot \cos x\, dx$$

$u = \sin x$

$du = \cos x\, dx$

$\frac{du}{\cos x} = dx$

$\int u^4 \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$\frac{1}{5} u^{4+1=5} + C \Rightarrow \frac{1}{5} \sin^5 x + C$

$$\int e^{2x} dx$$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

$\int e^u \cdot \frac{du}{2} = \frac{1}{2} \int e^u du$

$\frac{1}{2} e^u + C$

$\frac{1}{2} e^{2x} + C$

$$\int 5xe^{-x^2} dx$$

$u = -x^2$

$du = -2x\, dx$

$\frac{du}{-2x} = dx$

$\int 5\cancel{x} \cdot e^u \cdot \frac{du}{-2\cancel{x}} = -\frac{5}{2} \int e^u du$

$-\frac{5}{2} e^u + C$

$-\frac{5}{2} e^{-x^2} + C$

$$\int \sec^4 3x \tan 3x \, dx$$

$u = 3x$

$du = 3\,dx$

$\frac{du}{3} = dx$

$\int \sec^4 u \tan u \cdot \frac{du}{3}$

$\frac{1}{3} \int \sec^4 u \tan u \, du$

$\frac{1}{3} \int \sec^{\cancel{4}\,3} u \cdot \cancel{\tan u} \cdot \frac{dL}{\cancel{\sec u \tan u}}$

$\frac{1}{3} \int \sec^3 u \, dL$

$\frac{1}{3} \int L^3 \, dL = \frac{1}{3} \cdot \frac{1}{4} \cdot L^{3+1=4} + C$

$\frac{1}{12} \sec^4 3x + C$

$L = \sec u$

$dL = \sec u \tan u \, du$

$\frac{dL}{\sec u \tan u} = du$

$\frac{1}{3} \int \sec^4 u \tan u \, du$

$\frac{1}{3} \int L^4 \cancel{\tan u} \frac{dL}{\sec u \cancel{\tan u}}$

$\frac{1}{3} \int L^{\cancel{4}\,3} \cdot \frac{dL}{\cancel{L}} = \frac{1}{3} \int L^3 \, dL = \frac{1}{3} \cdot \frac{1}{4} \cdot L^{3+1=4} + C$

$\frac{1}{12} \sec^4 3x + C$

$$\int \frac{e^x}{e^x + 4} \, dx$$

$u = e^x + 4$

$du = e^x \, dx$

$\frac{du}{e^x} = dx$

$\int \frac{\cancel{e^x}}{u} \cdot \frac{du}{\cancel{e^x}} = \int \frac{1}{u} \, du$

$\ln|u| + C = \ln|e^x + 4| + C = \ln(e^x + 4) + C$

$$\int \frac{1}{8-2x}\,dx$$

$u = 8 - 2x$

$du = -2\,dx$

$\frac{du}{-2} = dx$

$$\int \frac{1}{u} \cdot \frac{du}{-2} = \frac{-1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln|u| + c = -\frac{1}{2} \ln|8-2x| + c$$

$$-\frac{1}{2} \ln|8-2x| + c$$

$$-\ln\left|\sqrt{8-2x}\right| + c$$

Example 8

$$\int \frac{x+1}{x^2+2x}\,dx$$

$u = x^2 + 2x$

$du = (2x+2)\,dx$

$\frac{du}{2x+2} = dx$

$$\int \frac{x+1}{u} \cdot \frac{du}{2x+2}$$

$$\int \frac{\cancel{x+1}}{u} \cdot \frac{du}{2(\cancel{x+1})} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + c$$

$$\frac{1}{2} \ln|x^2+2x| + c = \ln\left|\sqrt{x^2+2x}\right| + c$$

$$\frac{dy}{dx} = \frac{1}{x \ln x}$$

$u = \ln x$

$du = \frac{1}{x} dx$

$x\,du = dx$

$$\int dy = \int \frac{1}{x \ln x} dx$$

$$y = \int \frac{1}{x \ln x} dx$$

$$y = \int \frac{1}{\cancel{x} \cdot u} \cdot \cancel{x} du = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$

Let u be a differentiable function of x.

1) $\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$

$$\frac{d}{dx}[a^u] = \ln a \cdot a^u$$

2) $\int a^u \, du = \left(\frac{1}{\ln a}\right) a^u + C$ Use this rule when you do **u-substitution**

More Integrals

- $\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$
- $\int \sec u \, du = \ln|\sec x + \tan x| + C$
- $\int \cot u \, du =$ (left for an exercise for student)
- $\int \csc u \, du =$(left for an exercise for student)

1) $\int 5^{2x} \, dx$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

$= \int 5^{u} \cdot \frac{du}{2}$

$\frac{1}{2} \int 5^{u} du$

$\int a^u \, du = \left(\frac{1}{\ln a}\right) a^u + C$

$\frac{1}{2} \cdot \frac{1}{\ln 5} \cdot 5^{2x} + C$

$\frac{1}{\ln 25} \cdot 5^{2x} + C$

2) $\int \sin 2x \, 5^{\cos 2x} \, dx$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

$\int \sin u \, 5^{\cos u} \, dx$

$\int \sin u \cdot 5^{\cos u} \cdot \frac{du}{2}$

$L = \cos u$

$dL = -\sin u \, du$

$\frac{dL}{-\sin u} = du$

$\frac{1}{2} \int \cancel{\sin u} \cdot 5^{L} \cdot \frac{dL}{-\cancel{\sin u}}$

$-\frac{1}{2} \int 5^{L} dL = -\frac{1}{2} \cdot \frac{1}{\ln 5} \cdot 5^{\cos u}$

$\frac{-1}{2 \ln 5} \cdot 5^{\cos 2x} + C$

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = -\ln|\csc u + \cot u| + C$$

On the quiz, you will be asked to *derive*
an u, cot u, sec u, or csc u

$u = 5x$
$du = 5dx$
$\frac{du}{5} = dx$

2. $\int 10\cot 5x\,dx$

$\int 10 \cot u \cdot \frac{du}{5} = 2\int \cot u\,du = 2\ln|\sin u| + C$

$2\ln|\sin 5x| + C$

Recall that from the rules for differentiation,

Derivatives of an Inverse Trigonometric Function

1) $\dfrac{d}{dx}[arc\sin u] = \dfrac{u'}{\sqrt{1-u^2}}$

2) $\dfrac{d}{dx}[arc\cos u] = \dfrac{-u'}{\sqrt{1-u^2}}$

3) $\dfrac{d}{dx}[arc\tan u] = \dfrac{u'}{1+u^2}$

4) $\dfrac{d}{dx}[arc\cot u] = \dfrac{-u'}{1+u^2}$

5) $\dfrac{d}{dx}[arc\sec u] = \dfrac{u'}{|u|\sqrt{u^2-1}}$

6) $\dfrac{d}{dx}[arc\csc u] = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

Note of caution:

- Notice that in the inverse trig integrals "u" is different than u-substitution.

THEOREM 5.17 INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

xample

$$\int \frac{dx}{\sqrt{e^{2x}-1}}$$

$e^x = u$

$e^x \cdot e^x = u^2 = e^{2x}$

$e^x\,dx = du$

$dx = \dfrac{du}{e^x}$

$(e^x)^2 - 1^2$

$$\int \frac{dx}{\sqrt{u^2-1^2}} = \int \frac{1}{\sqrt{u^2-1^2}} \cdot \frac{du}{e^x}$$

$a=1$

$$\int \frac{du}{u\sqrt{u^2-1^2}} = \frac{1}{1}\operatorname{arcsec}\frac{|u|}{1} + C$$

$\operatorname{arcsec} \frac{e^x}{1} + c$

$\operatorname{arcsec} |e^x| + c$